Routing Games in the many players regime

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Outline

Background and motivation

- Routing games: model congestion when selfish players distribute their traffic demands on links/roads
- Applies to road congestion and road pricing, and selfish routing in data networks
- Nash equilibrium: (possible) outcome of repeated interactions of a finite number of players
- Wardrop equilibrium: equilibrium for an infinite number of players, actions of isolated users have no impact on the outcome
- The purpose of this work is to show the convergence of Nash Equilibrium to the Wardrop one when the number of players grows to infinity

Related work

- Analysis of routing games and their Nash equilibria ([1])
- Demonstration of the convergence of the Nash equilibrium to the Wardrop equilibrium using diagonal strict convexity assuming light traffic ([2])
- Particular case of polynomial costs ([3])

[1] Ariel Orda, Raphael Rom, and Nahum Shimkin. "Competitive routing in multiuser communication networks". In: *IEEE/ACM Trans. Netw.* 1 (5 1993), pp. 510–521.

[2] A. Haurie and P. Marcotte. "On the relationship between Nash-Cournot and Wardrop equilibria". In: *Networks* 15.3 (1985), pp. 295–308.

[3] E. Altman et al. "Competitive routing in networks with polynomial costs". In: *Automatic Control, IEEE Transactions on* 47.1 (Jan. 2002), pp. 92 –96.

The model, general case

A routing game is defined by:

- A directed graph $G = (\mathcal{N}, \mathcal{L})$, $\mathcal{N}$ nodes and $\mathcal{L}$ directed arcs
- A set W of source-destination pairs
- $\mathcal{I} = \{1, ..., I\}$ traffic classes , each defined by:
  - $w \in W$ a source-destination pair
  - $d_w \geq 0$ a traffic demand
  - R_w available paths between the source-destination pair w

Each player controls the repartition of its traffic demand among available paths:

- h^i_{wr} flow of player i over path r
- h_{wr} total flow over path r
- x^i_l flow of player i on link l
- $x_l = \sum_{i \in \mathcal{N}} x^i_l$ total flow over link l

The model, general case (cont'd)

We write the flow conservation equations:

$$\sum_{r \in R_w} h^i_{wr} = d^i_w, \quad w \in W, \tag{1}$$

$$\sum_{w \in W} \sum_{r \in R_w} h^i_{wr} \delta^l_{wr} = x^i_l, \quad l \in \mathcal{L}, \tag{2}$$

$$x^i_l \geq 0, \quad l \in \mathcal{L}, \tag{3}$$

with $\delta^l_{wr} = 1$ when link l is present on route $r \in R_w$ and 0 otherwise

The model, link routing

Link routing: the incoming traffic at each node can be split among the outgoing links. The flow conservation equations become:

$$r_v^i + \sum_{j \in In(v)} x_j^i = \sum_{j \in Out(v)} x_j^i \tag{4}$$

with:

$$r_v^i = \begin{cases} d_i & \text{, if } v \text{ is the source of player } i \\ -d_i & \text{, if } v \text{ is the destination of player } i \\ 0 & \text{, otherwise} \end{cases} \tag{5}$$

Player i controls its flow on every link $\mathbf{x}^i = \{x_l^i, l \in \mathcal{L}\}$.

The Nash-Cournot game: cost structure

We assume the following cost structure:

- $J_l^i(\mathbf{x})$ cost of player i on link l
- The cost is additive over links: $J^i(\mathbf{x}) = \sum_l J_l^i(\mathbf{x})$
- There exists a positive, strictly increasing, convex and continously differentiable cost density $t_l(x_l) \geq 0$ such that $J_l^i(x_l^i, x_l) = x_l^i t_l(x_l)$.

The case of atomless players: Wardrop

Wardrop equilibrium: the flow on every route serving an origin-destination pair is either zero, or its cost is equal to the minimum cost on that origin-destination pair.

$$h_{wr}(c_{wr} - \lambda_w) = 0, r \in R_w, w \in W, \tag{6}$$

$$c_{wr} - \lambda_w \geq 0, r \in R_w, w \in W, \tag{7}$$

$$\sum_{r \in R_w} h_{wr} = d_w, w \in W \tag{8}$$

with c_{wr} the total cost over the path $r \in R_w$.

The case of atomless players: Beckmann transformation

The Wardrop equilibrium reduces to an optimization problem, known as the Beckmann transformation ([4]):

$$\min_{\mathbf{x}} f(\mathbf{x}) = \sum_{l \in \mathcal{L}} \int_0^{\sum_{i \in \mathcal{N}} x_l^i} t_l(x)dx \tag{9}$$

subject to the flow conservation.

[4] Martin J. Beckmann, C. B. McGuire, and C. B. Winsten. *Studies in the Economics of Transportation*. Yale University Press, 1956.

Properties of Nash equilibrium

Important property: two symmetrical players behave the same way at a Nash equilibrium.

Lemma

Assume that players i and j have the same demand, source-destination pair and cost functions. Consider an equilibrium flow $\mathbf{x}$. Then for every link l, $x_l^i = x_l^j$.

Convergence to Wardrop equilibrium: main result

Theorem

The Nash equilibrium converges to the Wardrop equilibrium, in the following senses:

- *Let* $\mathbf{x}^m$ *be an equilibrium that corresponds to the replacement of each player* i *by* m *symmetrical copies. Then any limit of a converging subsequence is a Wardrop equilibrium*
- *The Wardrop equilibrium is an* ϵ*-equilibrium for the* m*-th game for all* m *large enough (i.e. no player can gain more than* ϵ *by deviating)*
- *For all* m *large enough, an equilibrium in the* m*-th game is an* ϵ*-Wardrop equilibrium*

Convergence to Wardrop equilibrium: sketch of proof

- We replace each player i by m identical sub-players, sharing equally the demand of i
- We apply the fact that these m subplayers have the same flows in equilibrium, and player i minimises:

$$\sum_{l \in \mathcal{L}} \left(\frac{1}{m} x_l^i t_l(x_l) + \frac{m-1}{m} \int_0^{x_l} t_l(x) dx \right) \qquad (10)$$

- The previous expression converges to the Beckmann transformation, and the three assertions of the theorem are proven by applying the results of ([5]).

[5] Eitan Altman et al. "Approximating Nash Equilibria In Nonzero-Sum Games". In: *International Game Theory Review* 2.2-3 (2000), pp. 155–172.

Example of application

- For all links, we consider an M/M/1 model, with capacity C_l for link l
- The cost of a link is the corresponding delay

$$J_l^i(x_l^i, x_l) = \begin{cases} 0 & x_l^i = 0 \\ \dfrac{x_l^i}{C_l - x_l} & x_l^i > 0\,,\ x_l < C_l \\ +\infty & x_l^i > 0\,,\ x_l \geq C_l \end{cases} \tag{11}$$

- Our result shows convegence to the Wardop equilibrium, even without the assumption of light traffic used in previous works.

Conclusion

- Convergence of the Nash equilibrium to the Wardrop equilibrium as the number of players grows has been shown
- Extension of a previous result by Haurie and Marcotte, and convergence has been shown under more general convexity assumptions
- The result applies in particular for an M/M/1 link model