A Justification Of The Fluid Network Model Using Stochastic Geometry

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Abstract

An important topic in performance evaluation of wireless networks is the modeling of inter-cell interference, to predict the distribution of the Signal to Interference plus Noise Ratio (SINR) in the network. The classical hexagonal model is generally intractable and requires extensive numerical calculations. Two approaches have been shown to produce tractable, closed-form formulas: Poisson networks (the interfering Base Stations (BSs) locations form a Poisson process) and fluid networks (the interfering BSs are replaced by a continuum of infinitesimal interferers). Compared to network measurements, the fluid model is known to be optimistic, while the Poisson model is pessimistic. We show that fluid networks are equivalent to dense Poisson networks. We show a Central Limit Theorem (CLT)-like result: the difference of interference predicted by the two models is Gaussian for dense networks with a known mean and variance. These results provide a justification of the fluid model. Furthermore, there is an interesting duality: for dense networks, all results proven for Poisson networks hold for fluid networks and vice-versa.

I. Introduction

With the explosion of mobile traffic, accurate and tractable models for performance evaluation of mobile networks are needed. A key performance indicator of wireless networks consists in the distribution of the SINR. Indeed, the knowledge of this parameter allows an estimation of Quality of Service (QoS) metrics such as the outage probability, namely the quantity of users which do not achieve a target data rate.

As networks become denser, prediction of the SINR distribution requires a model for the inter-cell interference. The most basic model is the hexagonal model: interfering BSs are placed on a regular infinite hexagonal grid ([1]-[8]). The hexagonal model is reasonable for regular deployments, however it is known to be intractable and requires extensive numerical computations. Several numerical techniques have been developed such as: numerical integration in hexagonal networks [7], [8], Monte Carlo simulations [10], [9] and approximation of the distribution of the interference factor [11].

Tractability is important not only because it reduces the computation time, but most of all because it allows to understand the influence of the system parameters (e.g propagation exponent) on the performance.

An alternative model is the Poisson model: the locations of interfering BSs are random and follow a Poisson process on the plane ([2]). Surprisingly, the Poisson process is tractable with many closed-form formulas ([22]). Furthermore, it is shown that measurements from operational networks lie in-between the optimistic hexagonal model and the pessimistic Poisson model.

In the search for tractable models, another model is the fluid model [15] [12]: the interfering BSs are replaced by infinitesimal ("fluid") interferers distributed in space. The fluid model is tractable and gives closed form formulas of the SINR distribution [13] [14]. Furthermore, the fluid model was shown to be very close to the intractable hexagonal model. Since the fluid model is a good approximation of the hexagonal model, it is optimistic as well. Fluid models are used in many fields of physical sciences. However, no physical justification for the fluid model is proposed, and the good fit between the fluid and hexagonal models did not find a satisfying explanation.

Our Contribution: In this article, we demonstrate that the fluid model is equivalent to the Poisson model for dense networks. Namely, the predicted SINR distribution is the same for the two models. We show a CLT-like result: the difference of interference predicted by the two models is Gaussian for dense networks with a known mean and variance. These results are a justification of the fluid model. Furthermore, there is an interesting duality: for dense networks, all results proven for Poisson networks hold for fluid networks and vice-versa.

The rest of the paper is organized as follows: In Section II we introduce the system model. In Section III we recall the main results of the fluid model. In Section IV we show that the fluid model is equivalent to the Poisson model for dense networks and that the difference of interference between the two models is Gaussian. In Section V, we compare the fluid and Poisson models numerically and show that they indeed coincide. Section VI concludes the paper. All proofs are presented as appendices.

II. System Model

We consider a wireless network in downlink. We focus on the performance of a single user. The multiplexing technology can be any access scheme in which the radio resources of a BS are divided in a number of parallel, orthogonal, non-interfering channels, i.e OFMDA (sub-carriers), TDMA (time slots), CDMA (codes, ignoring inter-code interference). Hence there is no intra-cell interference, only inter-cell interference. We will use the generic term of "resources" for sub-carriers, time slots or codes.

A. SINR of a user

We consider M BSs, transmitting at power P on each resource. We define $g_m(u)$ the propagation gain between BS m and user u on a resource. Interference is treated as Gaussian noise. The SINR γ_u of user u served by BS m on a resource is:

$$\gamma_u = \frac{P g_m(u)}{\sum\limits_{m' \neq m} P g_{m'}(u) + N_{th}}, \tag{1}$$

with N_{th} the thermal noise on a resource.

Treating each resource as an Additive White Gaussian Noise (AWGN) channel, the SINR received by a mobile allows to calculate the spectral efficiency D_u by using the Shannon formula:

$$D_u = \log_2(1 + \gamma_u). \tag{2}$$

We consider a path gain $g_m(u) = K r^{-\eta}$, where K is a constant, r is the distance between user u and BS m and η the path loss exponent.

B. From SINR to QoS

The knowledge of the SINR allows to calculate the throughput that may be reached by a user. The methods presented here allow to determine this characteristic with a high accuracy, for a user at any distance r from its serving BS. As a consequence, by using the Shannon relation, it becomes possible to determinate this characteristic, in a simple way, thanks to these methods. Moreover, since the throughput allows to know the quality of service that can be offered to a user, these methods allow to determine this characteristic with a high accuracy in a simple way. The minimum throughput, at the edge of the cell, can particularly be calculated. By doing an integration all over the cell range, the average throughput of the cell can be calculated, too.

III. Analytical fluid model network

The fluid model consists in replacing a given fixed finite number of transmitters by an equivalent continuous density of transmitters. Authors of [21] and [20] proposed a similar concept. The network is characterized as a density of interfering base BSs ρ_B [12], *i.e.* which use the same frequency bandwidth. The fluid model network approach, cf. [15] and [13], gives a closed-form formula of the interference factor [1] as a function of the location of a user. We remind hereafter the main results of this model as described in [15] and [12].

Considering a homogeneous network where all BSs transmit the same power we focus on a given cell and consider a round shaped network around this center cell with radius R_{nw} [15] [12]. The half distance between two base BSs is R_c (see Figure 1).

For a user at a distance r from its serving base BS, we can express the total interfering power $I(r)$ in terms of the special function Ψ introduced in appendix A:

[1] the interference factor is defined as the ratio between the power coming from the serving BS and the sum of other BSs powers received by a user, considering all BS transmitting the power P.

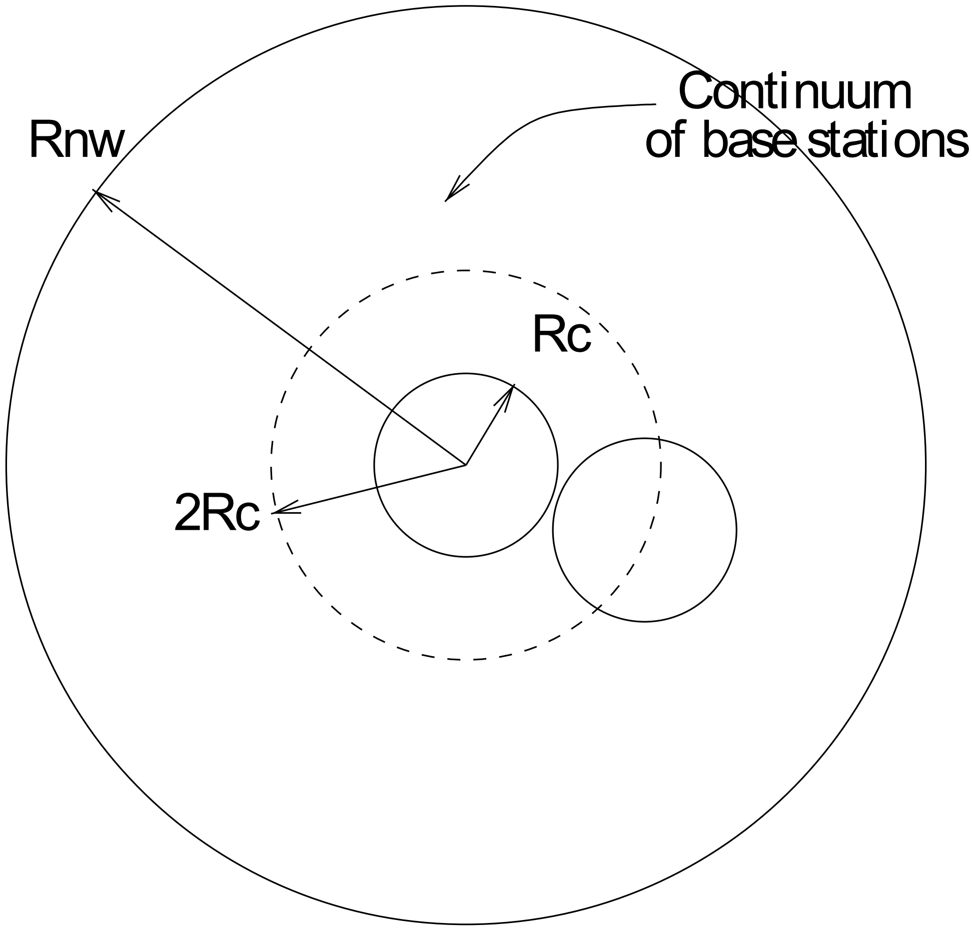

Figure 1. Network and cell of interest in the physical model; the distance between two BS is $2R_c$ and the network is made of a continuum of BSs.

$$I(r) = KP\rho_B\Psi(r,\eta). \tag{3}$$

We recall that Ψ can be approximated very accurately by a closed-form expression. Formula (3) holds for all distance values between BSs, and remains valid even when this distance reaches several kilometers. Indeed the fluid model network is developed whatever the density of BSs.

We can express the SINR (1) on a resource as:

$$\gamma_u = \frac{PKr^{-\eta}}{I(r) + N_{th}} \tag{4}$$

Neglecting the thermal noise, for a mobile located at a distance r from its serving BS, SINR formula is deduced from (4) and (3):

$$\gamma(r) = \frac{1}{r^{\eta}\rho_B\Psi(r,\eta)}. \tag{5}$$

IV. Main results

In this section we show that the fluid model can be justified by considering random networks, where the BSs positions follow a random process [4]. We show that there is a strong link between fluid networks and random networks, and that the two models become equivalent for dense networks. The required results on point processes are recalled in appendix .

A. Poisson networks

We now consider interfering BSs distributed on the plane according to a Poisson process of measure $\lambda(dx\, dy) = \lambda(x,y)dx\, dy$, with

$$\lambda(x,y) = \rho_B \text{ if } \sqrt{x^2+y^2} \in [2R_c, R_{nw}] \tag{6}$$

and $\lambda(x,y) = 0$ otherwise. Interfering BSs transmit at power P and we denote by $\{S_n\}_{n\in\mathbb{Z}}$ their locations. We denote $\mathcal{I}(r)$ the total interference received by a user at distance r from its serving BS. $\mathcal{I}(r)$ can be written as a shot noise:

$$\mathcal{I}(r) = \sum_{n\in\mathbb{Z}} \frac{KP}{\|r\vec{u}_r - S_n\|^\eta}, \tag{7}$$

with $\vec{u}_r$ a unit vector.

B. Statistics of the total interference power

The main results of our article are obtained by applying the results presented in appendix B. We obtain the mean interference, its variance and a CLT-like result.

Proposition 4.1:

$$\mathbb{E}\left[\mathcal{I}(r)\right] = KP\rho_B\Psi(r,\eta), \tag{8}$$

Proposition 4.2:

$$\mathbf{var}\mathcal{I}(r) = (KP)^2\rho_B\Psi(r,2\eta), \tag{9}$$

Proposition 4.3:

$$\frac{1}{\sqrt{\rho_B}}(\mathcal{I}(r) - I(r)) \underset{\rho_B\to+\infty}{\to} \mathcal{N}(0, \Psi(r,2\eta)(KP)^2)). \tag{10}$$

Proposition 4.1 states that the average interference power in the Poisson model is exactly the interference given by the fluid model. Proposition 4.2 gives a simple expression for the variance. In particular we have that the coefficient of variation of the interference vanishes for dense networks, since:

$$\frac{\mathbf{var}\mathcal{I}(r)}{\mathbb{E}\left[\mathcal{I}(r)\right]^2} = \frac{\Psi(r,2\eta)}{\rho_B\Psi(r,\eta)^2} \underset{\rho_B\to+\infty}{\to} 0. \tag{11}$$

Proposition 4.3 shows that the difference between the interference power predicted by the Poisson model and the fluid model becomes normally distributed with known variance. It is noted that all quantities are expressed in function of Ψ which can be approximated accurately by a closed form expression.

C. Coverage probability

As a corollary, we can calculate the coverage probability, i.e the probability that a user achieves a target SINR γ_0 to ensure adequate QoS. We consider a dense network, in which the interference at a given distance is normally distributed as stated by proposition 4.3. For a user located at a distance r from the serving BS, the coverage probability is:

$$p(r,\gamma_0) = \Phi\left(\frac{1}{KP\rho_B\Psi(r,2\eta)}(\rho_B\Psi(r,\eta) + \frac{N_{th}}{KP} - \frac{r^{-\eta}}{\gamma_0})\right), \tag{12}$$

with Φ the cumulative distribution function (c.d.f) of the normal distribution. The coverage probability for an uniformly distributed user:

$$(2/R_c^2)\int_0^{R_c} p(r,\gamma_0) r dr, \tag{13}$$

can be calculated by a one dimensional numerical integration.

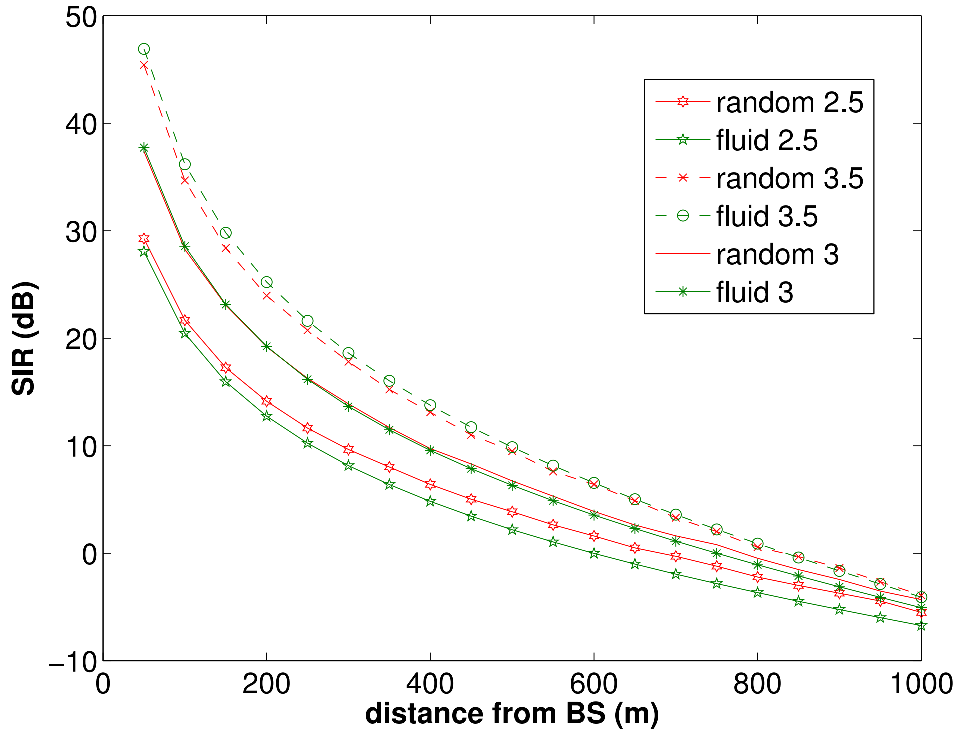

Figure 2. Mean SINR vs. distance to the BS; comparison of the random model to the fluid model network for $\eta \in \{2.5, 3, 3.5\}$.

V. Numerical experiments

In this section we compare results given by the fluid model to the ones given by the Poisson model. We show that approximating interference at a given location by a normally distributed random variable performs well numerically even for reasonable values of the density of interfering BSs.

At each location, we calculate the average value of the SINR given by the fluid model and the Poisson model. Simulation parameters are the following: $R_c = 1$ Km, $\eta \in \{2.5, 3, 3.5\}$ and $\rho_B = (3\sqrt{3}R_c^2/2)^{-1}$. We observe (Fig. 2) that the two approaches match very well for $\eta = 3$ and $\eta = 3.5$. For $\eta = 2.5$, the two approaches give very close results. However, the differences are more important, particularly for high distances from the BS.

A. C.d.f of the SINR

We compare the c.d.f of the SINR given by the Poisson model and the Gaussian approximation for the interference (formulas (12) and (13)). Fig.3 represents both quantities at a distance of 240m from the serving BS. The outage probability reaches 5% for a SINR of 10 dB and about 50% of users located at 240m have a SINR less than 12dB. 50% of users have a SINR comprised between 12 and 18dB. The exact distribution and the Gaussian approximation match with a good accuracy. It is recalled that for the Gaussian approximation, the distribution is known analytically (see IV-B). Only for high values of the c.d.f the curves are slightly different. From a practical point of view, this is not so important since the coverage probability involves mostly the low part of the c.d.f. Namely we want to determine which SINR can be achieved by *most of the users*, say 95%.

B. Impact of the density of BSs

We now analyze the impact of the density of interfering BS to show that the Gaussian approximation becomes more accurate when the network becomes denser. Fig 4 shows the c.d.f of SINR at a distance of 400m from the serving BS, for $\eta = 2.5$. When the number of transmitters increases from λ (left curve) to 10λ (right curve), the c.d.f indeed tends to the curve corresponding to equations (12) and (13). As expected, densification results in more inter-cell interference, so that the average SINR diminishes.

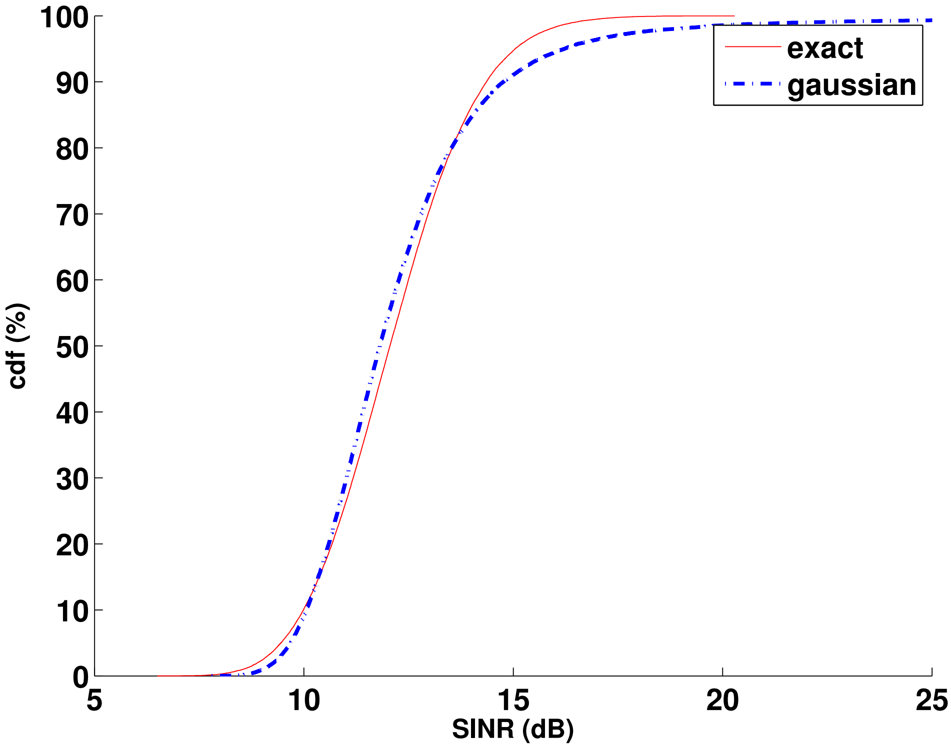

Figure 3. c.d.f of the SINR at a distance of 240 m to the BS for $\eta = 2.5$

C. Impact of pathloss exponent on the c.d.f

Fig. 5 shows the c.d.f of the SINR at a distance of 400m from the serving BS. The c.d.f of the SINR improves when the pathloss exponent η increases since it diminishes the amount of inter-cell interference. For $\eta = 2.5$, a SINR of -5 dB can be reached for an outage probability of 10%. For $\eta = 3.5$ (resp. 4), a SINR of 1 dB (resp.3 dB) can be reached for the same outage probability of 10%. We also observe that the Gaussian approximation (formulas (12) and (13)) becomes more accurate when the pathloss parameter decreases.

VI. Conclusion

We have established a link between two models for evaluating the performance of wireless networks: Poisson networks (the interfering BSs locations form a Poisson process) and fluid networks (the interfering BSs are replaced by a continuum of infinitesimal interferers). Both approaches have been shown to produce tractable, closed-form formulas, unlike the classical hexagonal model which is not tractable. We have shown that fluid networks are equivalent to dense Poisson networks. We have shown a CLT-like result: the difference of interference predicted by the two models is Gaussian for dense networks with a known mean and variance. These results provide a justification of the fluid model. Furthermore, there is an interesting duality: for dense networks, all results proven for Poisson networks hold for fluid networks and vice-versa.

Appendix A
A useful integral

In this article we use repeatedly the elliptic integral:

$$\Psi(r,\eta) = \int_0^{2\pi} \int_{2R_c}^{R_{nw}} (u^2 + r^2 - 2ur\cos(\theta))^{-\frac{\eta}{2}} u du d\theta. \tag{14}$$

for the interfering power. While Ψ has no simple closed-form expression, it is shown in [16] the interference calculation can be approximated by:

$$\Psi(r,\eta) \approx \frac{2\pi r^{\eta}}{\eta - 2}[2(R_c - r)^{2-\eta} - (R_{nw} - r)^{2-\eta}]. \tag{15}$$

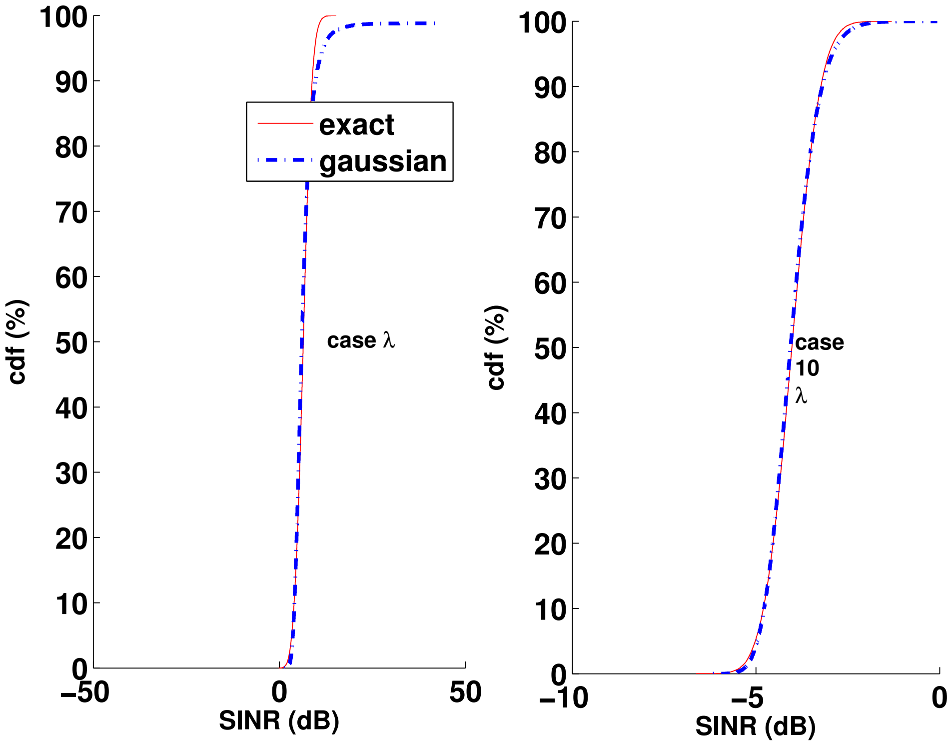

Figure 4. c.d.f of the SINR at a distance of 400 m to the BS for $\eta = 2.5$

In case do not want to use the analytical approximation, an efficient numerical solution would be to tabulate values of $r \mapsto \Psi(r, \eta)$ for several values of η. As seen in subsection IV-B, for a given value of η, the full c.d.f of the SINR can be calculated as soon as we know $r \mapsto \Psi(r, \eta)$ and $r \mapsto \Psi(r, 2\eta)$. Hence using Ψ as a tabulated function is simple and numerically efficient.

Appendix B
Poisson processes and shot noise

We recall results on point processes which can be found for instance in [23]. Consider λ a measure on $\mathbb{R}^2$ and $\{S_n\}_{n\in\mathbb{Z}}$ a random collection of points in $\mathbb{R}^2$. Given B a Borel set, we define

$$N(B) = \sum_{n\in\mathbb{Z}} \mathbf{1}_B(S_n), \tag{16}$$

the number of points contained in B. $\{S_n\}_{n\in\mathbb{Z}}$ is a Poisson process with measure λ if $N(B)$ is a Poisson random variable of parameter $\lambda(B)$ and $N(B)$, $N(B')$ are independent if
$B \cap B' = \emptyset$.

Consider $f : \mathbb{R}^2 \to \mathbb{R}$ a positive function, and define the random function:

$$F(r) = \sum_{n\in\mathbb{Z}} f(r\vec{u}_r, S_n), \tag{17}$$

with $\vec{u}_r$ a unit vector. F is called a *shot noise*, with *impulse response* f.

The average shot noise is obtained from the Campbell formula [3]:

$$\mathbb{E}\left[F(r)\right] = \int_{\mathbb{R}^2} f(r\vec{u}_r, u)\lambda(du) \tag{18}$$

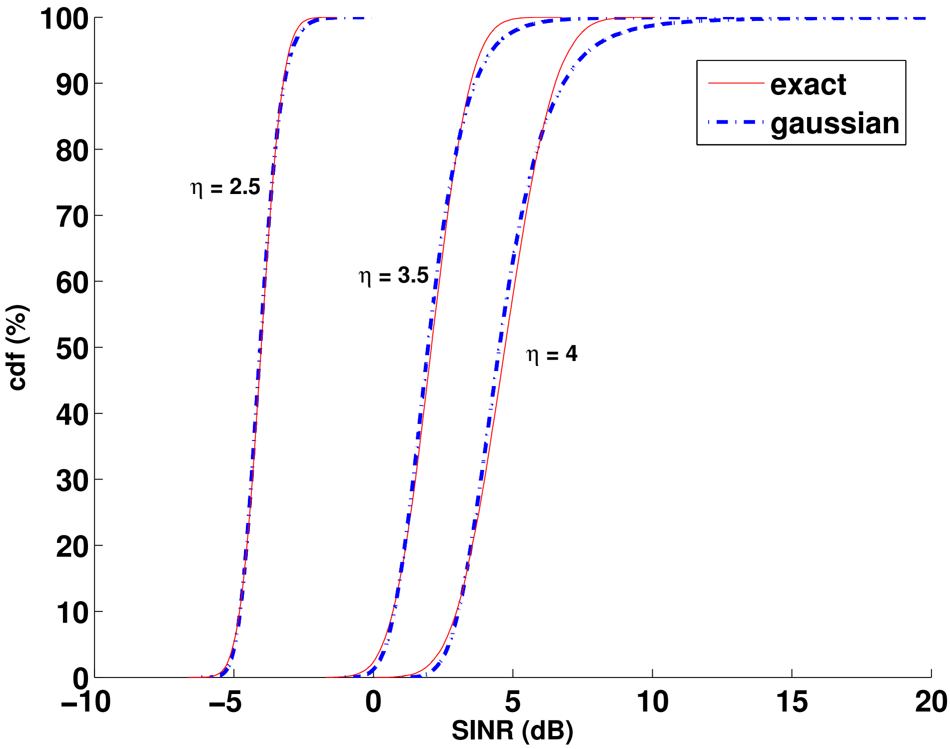

Figure 5. c.d.f of the SINR at a distance of 400 m to the BS

The second moment can be calculated using a second-order Campbell formula:

$$\begin{aligned}\mathbb{E}\left[F(r)^2\right] &= \int_{\mathbb{R}^2} f(r\vec{u}_r, u)^2 \lambda(du) \\ &+ \int_{\mathbb{R}^2\times\mathbb{R}^2} f(r\vec{u}_r, u) f(r\vec{u}_r, v)\lambda(du)\lambda(dv) \\ &= \int_{\mathbb{R}^2} f(r\vec{u}_r, u)^2 \lambda(du) + \mathbb{E}\left[F(r)\right]^2 \end{aligned} \tag{19}$$

We finally state a CLT for shot noise. Write λ as $\lambda(du) = \lambda_0 \lambda_1(du)$. Then

Proposition B.1:

$$\frac{1}{\sqrt{\lambda_0}}(F(r) - \mathbb{E}\left[F(r)\right]) \to \mathcal{N}(0, \sigma^2) \text{ in distribution,} \tag{20}$$

$$\sigma^2 = \int_{\mathbb{R}^2} f(r\vec{u}_r, u)^2 \lambda_1(du). \tag{21}$$

Proof: r will remain constant throughput the proof, and will sometimes be omitted for the sake of clarity. We recall that if $X \equiv Poisson(\lambda)$ then

$$\frac{X-\lambda}{\sqrt{\lambda}} \underset{\lambda\to+\infty}{\to} \mathcal{N}(0,1) \text{ in distribution.}$$

First assume that f is an indicator function, $f(r\vec{u}_r, u) = \mathbf{1}_B(u)$ with B a Borel set. Then $F(r) = N(B) \equiv Poisson(\lambda_0\lambda_1(B))$. Hence

$$\frac{1}{\sqrt{\lambda_0}}(F(r) - \mathbb{E}\left[F(r)\right]) \to \mathcal{N}(0, \lambda_1(B)).$$

Furthermore, if f is a simple function, $f(r\vec{u}_r, u) = \sum_{i=1}^{N} a_i \mathbf{1}_{B_i}(u)$, with $\cap_{i=1}^{N} B_i = \emptyset$, then

$$F(r) = \sum_{i=1}^{N} a_i N(B_i),$$

which proves that:

$$\frac{1}{\sqrt{\lambda_0}}(F(r) - \mathbb{E}\left[F(r)\right]) \to \mathcal{N}\left(0, \sum_{i=1}^{N} a_i^2 \lambda_1(B_i)\right)).$$

By a monotone class argument, for $u \mapsto f(r,u)$ measurable we obtain the result:

$$\frac{1}{\sqrt{\lambda_0}}(F(r) - \mathbb{E}\left[F(r)\right]) \to \mathcal{N}\left(0, \int_{\mathbb{R}^2} f(r\vec{u}_r, u)^2 \lambda_1(du)\right),$$

which concludes the proof. ■

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